ROOK THEORY, COMPOSITIONS, AND ZETA FUNCTIONS

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ABSTRACT. A new family of Dirichlet series having interesting combinatorial properties is introduced. Although they have no functional equation or Euler product, under the Riemann Hypothesis it is shown that these functions have no zeros in $\mathrm{Re}(s) > 1/2$. Some identities in the ring of formal power series involving rook theory and continued fractions are developed.

1. INTRODUCTION

One of the most mysterious functions in analysis and number theory is the Riemann Zeta function:

$$\zeta(s) = \prod_{i=1}^{\infty} \frac{1}{1 - q_i^{-s}}, \qquad \mathrm{Re}(s) > 1,$$

where $q_1 = 2, q_2 = 3$, is the sequence of prime numbers. In his great paper of 1859, Riemann conjectured that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1/2$. This is now known as the Riemann hypothesis (RH for short) and is perhaps the greatest unsolved problem in mathematics.

In the author's dissertation [Ha1], the coefficient of n^{-s} in $(\zeta(s) - 1)^k$ was shown to have an interpretation in terms of combinatorial *rook theory*. More recent work [Ha2] has led to the investigation of a set of Dirichlet series with peculiar "pseudo Euler products". The simplest of these is

$$\zeta_2(s) = \prod_{i=1}^{\infty} \cfrac{1}{1 - \cfrac{q_i^{-s}}{1 - \cfrac{q_{i+1}^{-s}}{1 - \cfrac{q_{i+2}^{-s}}{1 - \ldots}}}}, \qquad \mathrm{Re}(s) > 1.$$

Hoping to gain insight into the RH led the author to investigate the location of the zeros of the corresponding Dirichlet series, with the following result:

Theorem 1.1. a) $\zeta_2(s)$ *can be analytically continued to the line* $\mathrm{Re}(s) = 0$.
b) *If* $\zeta_2(s) = 0$, *then* $\zeta(ks) = 0$ *for some* $k \in \mathbb{Z}^+$. *Thus under the RH* $\zeta_2(s)$ *has no zeros in* $\mathrm{Re}(s) > 1/2$.

Although $\zeta_2(s)$ doesn't appear to satisfy a functional equation, combinatorially it has many similarities to $\zeta(s)$. For example,

$$\zeta_2(s)^{-1} = \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n^s},$$

where

$$\mu_2(n) = \begin{cases} 0 & \text{if } q_i^2 \mid n \text{ or } q_i q_{i+1} \mid n \text{ for some } i; \\ (-1)^{\omega(n)} & \text{else,} \end{cases}$$

with $\omega(n)$ the number of prime factors of n. This bears close resemblance to the well-known identity $\zeta(s)^{-1} = \sum_n \mu(n) n^{-s}$, where

$$\mu(n) = \begin{cases} 0 & \text{if } q_i^2 \mid n \text{ for some } i; \\ (-1)^{\omega(n)} & \text{else.} \end{cases}$$

One of the equivalent forms of the RH is that the function $M(x)$ (defined as $\sum_{n\leq x} \mu(n)$), is $O(x^{1/2+\epsilon})$ $\forall \epsilon > 0$ [Tic, p.370]. We show this implies the same estimate holds for $\sum_{n\leq x} \mu_2(n)$:

Theorem 1.2. *Under the RH,*

$$\sum_{n\leq x} \mu_2(n) = O(x^{1/2+\epsilon}) \ \forall \epsilon > 0.$$

Some unanswered questions are whether or not the assumption $\sum_{n\leq x} \mu_2(n) = O(x^{1/2+\epsilon})$ $\forall \epsilon > 0$ implies the RH, and if the line $\mathrm{Re}(s) = 0$ forms a natural boundary for $\zeta_2(s)$.

Actually Theorems 1.1 and 1.2 have much more general versions, that hold for any of our class of Dirichlet series (which arise from more complicated continued fraction structures known as *branching continued fractions*, and which have an interpretation in terms of rook theory).

Remark on notation. q_i will always denote the ith largest prime, but many of the theorems in this paper also hold true if we replace q_i by some arbitrary permutation of the primes. So p_i will denote the ith member of some ordering of the primes $p_1, p_2, \ldots$, where each prime occurs once and only once in this list. Throughout the article σ denotes $\mathrm{Re}(s)$. LHS and RHS are abbreviations for left hand side and right hand side, respectively.

2. COMBINATORIAL PROPERTIES

In this section we define the functions $\zeta_r(s)$ and list the combinatorial identities which they satisfy, putting off the proofs until section 6. For $r \geq 1$, set

$$\zeta_r(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \prod_{i\geq 1} \binom{\nu_i + \nu_{i-1} + \ldots + \nu_{i-r+1}}{\nu_i}, \qquad n = \prod_i p_i^{\nu_i}.$$

For example,

$$\zeta_2(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \prod_{i\geq 1} \binom{\nu_i + \nu_{i-1}}{\nu_i}.$$

Note that $\zeta_1(s) = \zeta(s)$, and that different orderings of the primes $p_1, p_2, \ldots$ yield different Dirichlet series for $r > 1$ but not for $r = 1$. If $s = \sigma + it$,

$$| \zeta_r(s) | \le \zeta_r(\sigma) \le \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \prod_{i \ge 1} \binom{\nu_i + \nu_{i-1} + \ldots + \nu_1}{\nu_i}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \binom{\sum_i \nu_i}{\nu_1, \nu_2, \ldots} = \frac{1}{1 - \sum_p p^{-\sigma}}$$

provided $\sigma > \beta$, where β is the real solution to the equation $\sum_p p^{-\beta} = 1$. Note that $\beta < 2$. Thus $\zeta_r(s)$ converges for $\sigma > \beta_r$, for some β_r satisfying $1 \le \beta_r \le \beta$ (section 3 contains more on β_r). All of the combinatorial properties of $\zeta_r(s)$ listed in this section hold for sure if $\sigma > \beta$. Some, such as the product identity for ζ_r below, hold for $\sigma > 1$.

Pseudo Euler product. As mentioned earlier, $\zeta_r(s)$ has a factorization, into generalized continued fractions, for $r > 2$ as well. For example,

$$\zeta_3(s) = \prod_{i=1}^{\infty} \frac{1}{1 - \dfrac{p_i^{-s}}{(1 - \dfrac{p_{i+1}^{-s}}{(1 - \frac{p_{i+2}^{-s}}{1 - \ldots})(1 - \frac{p_{i+3}^{-s}}{1 - \ldots})})(1 - \dfrac{p_{i+2}^{-s}}{(1 - \frac{p_{i+3}^{-s}}{1 - \ldots})(1 - \frac{p_{i+4}^{-s}}{1 - \ldots})})}}.$$

Here each term branches into 2 new terms, and in general $\zeta_r(s)$ has a product where each term branches into $r - 1$ new terms

$$\zeta_r(s) = \prod_{i=1}^{\infty} \frac{1}{1 - \dfrac{p_i^{-s}}{(1 - \dfrac{p_{i+1}^{-s}}{(1 - \frac{p_{i+2}^{-s}}{1 - \ldots}) \cdots (1 - \frac{p_{i+r}^{-s}}{1 - \ldots})}) \cdots (1 - \dfrac{p_{i+r-1}^{-s}}{(1 - \frac{p_{i+r}^{-s}}{1 - \ldots}) \cdots (1 - \frac{p_{i+2r-2}^{-s}}{1 - \ldots})})}}.$$

It is easy to describe the coefficient of n^{-s} in $\zeta_r(s)^{-1}$. Define $\mu_r(n)$ as follows:

$$\mu_r(n) = \begin{cases} 0 & \text{if } p_i p_j \mid n \text{ for some } i, j \text{ with } | i - j | \le r - 1; \\ (-1)^{\omega(n)} & \text{else.} \end{cases}$$

For example, if $p_1 = 2, p_2 = 3$, et cetera, then $\mu_2(21) = 1$, but $\mu_3(21) = 0$ since $p_2 p_4 \mid 21$. Note that $\mu_1(n) = \mu(n)$.

Theorem 2.1.

$$\zeta_r(s)^{-1} = \sum_{n=1}^{\infty} \frac{\mu_r(n)}{n^s}.$$

Corollary 2.2. *$\zeta_r(s)$ has an analytic continuation to the line $\sigma = 1$, and is zero free for $\sigma > 1$.*

Proof of the Corollary. Since $\sum_n \mu_r(n) n^{-s}$ converges absolutely and hence uniformly on compact subsets of $\sigma > 1$, it represents an analytic function throughout that region. This function agrees with $\zeta_r(s)^{-1}$ for $\sigma > \beta$, so by uniqueness of analytic continuation also agrees with $\zeta_r(s)^{-1}$ for $\sigma > 1$. Since $| \zeta_r(s)^{-1} | < \infty$ in this region, $\zeta_r(s) \ne 0$ for $\sigma > 1$. □

Some other combinatorial properties of interest:

$$\zeta_r(s)^x = \sum_{n=1}^{\infty} \frac{1}{n^s} \prod_{i \geq 1} \binom{\nu_i + \nu_{i-1} + \ldots + \nu_{i-r+1} + x - 1}{\nu_i}, \qquad x \in \mathbb{C}, n = \prod_i p_i^{\nu_i},$$

and

$$\zeta_r(s) = \exp(\sum_{n=2}^{\infty} \frac{1}{n^s} \frac{1}{\nu_b} \prod_{i \geq b+1} \binom{\nu_i + \nu_{i-1} + \ldots + \nu_{i-r+1} - 1}{\nu_i}), \qquad n = \prod_i p_i^{\nu_i},$$

where $b = b(n) = \min_{\nu_i > 0} i$. The first identity holds as long as $\zeta_r(\sigma) < 2$.

3. Domain of Convergence

In section 2 we mentioned that the series representation for $\zeta_r(s)$ converges for $\sigma > \beta$. In this section we show $\zeta_2(s)$ converges for $\sigma > 1$, assuming we are dealing with the canonical ordering of primes $q_1 = 2, q_2 = 3$, et cetera. Using the trivial estimate $\binom{n}{k} \leq 2^n$, we have

$$\zeta_2(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \prod_{i \geq 1} \binom{\nu_i + \nu_{i-1}}{\nu_i}$$

$$= \sum_{a,b,c,d,e,\ldots} \frac{\binom{a+b}{b} \binom{b+c}{c} \binom{c+d}{d} \binom{d+e}{e} \binom{e+f}{f} \cdots}{2^{a\sigma} 3^{b\sigma} 5^{c\sigma} 7^{d\sigma} 11^{e\sigma} 13^{f\sigma \ldots}}$$

$$< \sum_{a,b,c,d,e,\ldots} \frac{\binom{a+b}{b} \binom{b+c}{c} \binom{c+d}{d} 2^{d+e} 2^{e+f} 2^{f+g} \cdots}{2^{a\sigma} 3^{b\sigma} 5^{c\sigma} 7^{d\sigma} 11^{e\sigma} 13^{f\sigma \ldots}}$$

$$\leq \sum_{a,b,c,d} \frac{\binom{a+b}{b} \binom{b+c}{c} \binom{c+d}{d} 2^d}{2^{a\sigma} 3^{b\sigma} 5^{c\sigma} 7^{d\sigma}} \prod_{p \geq 11} \frac{1}{1 - 4/p^{\sigma}}.$$

The sum on the right can be evaluated by setting $r = 2, x = -1, x_1 = -1/2^{\sigma}, x_2 = -1/3^{\sigma}, x_3 = -1/5^{\sigma}, x_4 = -2/7^{\sigma}$, and $x_j = 0$ for $j > 4$ in Theorem 6.2 (proved in section 6). The result is that for $\sigma > \beta$,

$$\sum_{a,b,c,d} \frac{\binom{a+b}{b} \binom{b+c}{c} \binom{c+d}{d} 2^d}{2^{a\sigma} 3^{b\sigma} 5^{c\sigma} 7^{d\sigma}}$$

$$= \frac{1}{1 - (\frac{1}{2^{\sigma}} + \frac{1}{3^{\sigma}} + \frac{1}{5^{\sigma}} + \frac{2}{7^{\sigma}} - \frac{1}{10^{\sigma}} - \frac{2}{14^{\sigma}} - \frac{2}{21^{\sigma}})}.$$

If the LHS of this equation were to approach a pole, the denominator of the RHS would have to approach zero. Graphing it on a computer, we find that the denominator is positive for $\sigma > 1$ (and first becomes zero around $\sigma = .97$), and so the

LHS converges for $\sigma > 1$. Since

$$\zeta_2(\sigma) \leq \sum_{a,b,c,d} \frac{\binom{a+b}{b}\binom{b+c}{c}\binom{c+d}{d}2^d}{2^{a\sigma}3^{b\sigma}5^{c\sigma}7^{d\sigma}} \prod_{p \geq 11} \frac{1}{1-4/p^\sigma},$$

and since the product on the right converges for $\sigma > 1$, we see that the domain of convergence for $\zeta_2(s)$ is $\sigma > 1$.

For $r > 2, \zeta_r(\sigma)$ does not converge for all $\sigma > 1$ since

$$\zeta_r(\sigma) < \sum_{a,b,c} \frac{\binom{a+b}{b}\binom{a+b+c}{c}}{2^{a\sigma}3^{b\sigma}5^{c\sigma}} = \frac{1}{1-(1/2^\sigma+1/3^\sigma+1/5^\sigma)}$$

which has a pole in $\sigma > 1$ since $1/2 + 1/3 + 1/5 > 1$. For some values of r, a better upper bound than β for the domain of convergence of $\zeta_r(s)$ can be obtained by extending the argument given above for $\zeta_2(s)$.

4. ANALYTIC CONTINUATION

Theorem 4.1. *Set*

$$C_i(s) = \cfrac{1}{1 - \cfrac{p_i^{-s}}{1 - \cfrac{p_{i+1}^{-s}}{1 - \cfrac{p_{i+2}^{-s}}{1 - \ldots}}}}, \qquad \sigma > 1.$$

Then $C_i(s)$ can be analytically continued to the line $\sigma = 0$.

Proof. First a lemma:

Lemma 4.2. *For any $m, N \in \mathbb{Z}^+$*

$$\sum_{\substack{n \\ \nu(n)=m \\ b(n)=N}} D_2(n) \leq 8^m,$$

where

$$D_2(n) = \frac{1}{\nu_b} \prod_{j=b+1}^{\infty} \binom{\nu_j + \nu_{j-1} - 1}{\nu_j},$$

$\nu(n) = \sum_i \nu_i$, and $b = b(n) = \min_{\nu_i > 0} i$ as before.

Proof. Note that for any $k > N$, if $\nu_k > 0$ and $\nu_{k-1} = 0$ then $\binom{\nu_j + \nu_{j-1} - 1}{\nu_j} = 0$ and so $D_2(n) = 0$. Thus we need only consider integers of the form $p_N^{\nu_N} p_{N+1}^{\nu_{N+1}} \cdots p_k^{\nu_k}$, where $\nu_j > 0$ for $N \leq j \leq k$, with k in the range $N \leq k \leq N + m - 1$. We have

$$\sum_{\substack{n \\ \nu(n)=m \\ b(n)=N}} \frac{1}{\nu_N} \prod_{j=N+1}^{\infty} \binom{\nu_j + \nu_{j-1} - 1}{\nu_j} \leq \sum_{\substack{n \\ \nu(n)=m,b(n)=N \\ \nu_j>0, N \leq j \leq k}} \frac{1}{\nu_N} \prod_{j=N+1}^{\infty} 2^{\nu_j + \nu_{j-1}}$$

$$\leq 4^m 2^m = 8^m,$$

since 2^m is an upper bound for the number of solutions to $\nu_N + \nu_{N+1} + \ldots + \nu_k = m, \nu_j > 0$ for $N \leq j \leq k$. $\qquad \square$

JAMES HAGLUND

Continuing with the proof of Theorem 4.1, fix $\delta > 0$, and choose N big enough so that $j \geq N$ implies $p_j^{\delta} > 9$ and also $N \geq i$. By Corollary 6.6, for σ sufficiently large,

$$C_N(s) = \exp(\sum_{\substack{n \\ b(n)=N}} \frac{1}{n^s} D_2(n)).$$

By Lemma 4.2,

$$\left| \sum_{\substack{n \\ b(n)=N}} \frac{1}{n^s} D_2(n) \right| \leq \sum_{m=1}^{\infty} \sum_{\substack{n \\ b(n)=N \\ \nu(n)=m}} \frac{1}{9^m} D_2(n)$$

$$\leq \sum_{m=1}^{\infty} \frac{8^m}{9^m} < \infty, \qquad \sigma > \delta.$$

Hence $C_N(s)$ is the exponential of a bounded function in $\sigma > \delta$. Since $C_i(s)$ can be expressed as a rational function in $C_N(s)$—for example,

$$C_1(s) = \frac{1}{1 - C_2(s)p_1^{-s}}$$

—we see that $C_i(s)$ is a meromorphic function in $\sigma > \delta$ for any $\delta > 0$ and thus for $\sigma > 0$. □

Remark. This proof can easily be modified to show that the branching continued fractions occurring in the product formula for $\zeta_r(\mathrm{s})$ can also be continued to the line $\sigma = 0$. It is not clear whether they can be extended beyond $\sigma = 0$.

Theorem 4.3. *Assume that $C_i(s) = \exp(g(s))$, where $g(\mathrm{s})$ is bounded in $\sigma > \delta$. Then $\prod_{j=1}^{i} C_j(s)$ is zero free in $\sigma > \delta$.*

Proof. If $i = 1$, the product in question is the exponential of a bounded function and is thus zero free. So assume $i > 1$. As is well-known, the finite continued fraction

$$\cfrac{1}{1 - \cfrac{x_1}{1 - \cfrac{x_2}{\cdots 1 - \cfrac{x_{n-1}}{1 - x_n}}}}$$

can be written as

$$\frac{E_2(-x_2, -x_3, \ldots, -x_n)}{E_2(-x_1, -x_2, \ldots, -x_n)},$$

where E_2 is a certain polynomial (the exact definition of E_2 follows Theorem 6.1). Note that for $j \leq i - 1$, $C_j(s)$ can be written as the finite continued fraction

$$\cfrac{1}{1 - \cfrac{p_j^{-s}}{1 - \cfrac{p_{j+1}^{-s}}{\cdots 1 - \cfrac{C_i(s)}{p_{i-1}^s}}}}.$$

Hence

$$\prod_{j=1}^{i} C_j = C_i(s) \prod_{j=1}^{i-1} \frac{E_2(-p_{j+1}^{-s}, -p_{j+2}^{-s}, \ldots, -C_i(s)p_{i-1}^{-s})}{E_2(-p_j^{-s}, -p_{j+1}^{-s}, \ldots, -C_i(s)p_{i-1}^{-s})}$$

$$= \frac{C_i(s)}{E_2(-p_1^{-s}, -p_2^{-s}, \ldots, -C_i(s)p_{i-1}^{-s})}.$$

Since $C_i(s)$ is zero free and bounded for $\sigma > \delta$, the theorem follows. $\qquad \square$

Call a permutation $p_1, p_2, \ldots$ of the primes *bounded* if $p_j = q_i$ implies $|j - i| \leq B$ for some constant B and all $i, j \geq 1$.

Theorem 4.4. *Let $p_1, p_2, \ldots$ be a bounded permutation of the primes. Then, given $\delta > 0$,*

$$\zeta_2(s) = \exp(g(s)) \prod_{i=1}^{N} C_i(s) \prod_{k=1}^{M} \zeta(ks)^{a_k}$$

for some integers N, M which approach ∞ as $\delta \to 0$. The numbers a_k are positive integers and $g(\mathrm{s})$ is bounded in $\sigma > \delta$ (N, M, and g depend on δ but the a_k do not).

Before proving this we list some consequences of Theorems 4.1, 4.3, and 4.4.

Corollary 4.5. *By the analytic continuation of $\zeta(s)$, $\zeta_2(s)$ extends to a meromorphic function in $\sigma > 0$; if $\zeta_2(s) = 0$, then $\zeta(ks) = 0$ for some $k \in \mathbb{Z}^+$.*

Corollary 4.6. *$\zeta_2(s)$ has a pole at $s = 1/k, k \in \mathbb{Z}^+$.*

Since $s = 0$ is a limit point of poles, it is an essential singularity of $\zeta_2(s)$. Similarly, any point on the line $\sigma = 0$ is a limit point of zeros of the term

$$\prod_{k=1}^{M} \zeta(ks)^{a_k}$$

as $M \to \infty$, but some or all of these zeros could conceivably be cancelled by poles of $C_i(s)$ for some i, and the question of whether or not the line $\sigma = 0$ is a natural boundary of $\zeta_2(s)$ remains open.

Proof of Theorem 4.4. The basic idea is to approximate $C_i(s)$ for large i by

$$\cfrac{1}{1 - \cfrac{p_i^{-s}}{1 - \cfrac{p_i^{-s}}{1 - \cfrac{p_i^{-s}}{\cdots}}}},$$

and show that

$$\prod_{i > N_0} \frac{1}{1 - \dfrac{p_i^{-s}}{1 - \dfrac{p_i^{-s}}{\cdots}}}$$

is approximately

$$\prod_{i > N_0} \prod_{k=1}^{M} \frac{1}{(1 - p_i^{-ks})^{a_k}},$$

which is a product of ζ functions modulo some bounded terms.

Fix $\delta > 0$ and choose an integer $B \in \mathbb{Z}^+$ such that $p_j = q_i \implies |j - i| \le B$. Next choose N_0 so that $N_0 \ge B + 1, (q_{N_0 - B})^{\delta} > 8$, and for $j > N_0, C_j(s) = \exp(f(s))$ with $f(s)$ bounded in $\sigma > \delta$ (as we did in Theorem 4.1). For σ sufficiently large,

$$(1) \qquad \zeta_2(s) = \prod_{i=1}^{N_0} C_i(s) \exp\Big(\sum_{\substack{m,i \\ m \ge 1, i > N_0}} \frac{1}{m p_i^{ms}} + \sum_{\substack{\omega(n) \ge 2 \\ b(n) > N_0}} \frac{D_2(n)}{n^s} \Big).$$

Choose $N_1 > 1/\delta$. By Lemma 4.2,

$$\left| \sum_{\substack{\omega(n) \ge 2 \\ \nu(n) \ge N_1 \\ b(n) > N_0}} \sum_{n} \frac{D_2(n)}{n^s} \right| \le \sum_{k = N_0 + 1}^{\infty} \sum_{m = N_1}^{\infty} \frac{8^m}{(q_{k-B})^{m\delta}}$$

(since $b(n) = k \implies$ all primes dividing n are $\ge q_{k-B}$)

$$= \sum_{k = N_0 + 1}^{\infty} \Big(\frac{8}{q_{k-B}^{\delta}} \Big)^{N_1} \frac{1}{1 - \dfrac{8}{q_{k-B}^{\delta}}} < \infty$$

since $N_1 \delta > 1$ and $q_{N_0 - B}^{\delta} > 8$.

Next we consider the terms inside the exponential on the RHS of (1) satisfying $2 \le \omega(n), 1 < \nu(n) < N_1$. Let us start with $\nu(n) = 2$. The term in question is

$$\sum_{i > N_0} \frac{1}{p_i^s p_{i+1}^s},$$

which is approximately $\sum_{i > N_0} 1/(q_{i-B})^{2s} (\Sigma_i 1/q_i^s$ can be continued to the line $\sigma = 0$, where it has a natural boundary [Tic, p.215]). Thus

$$\left| \sum_{i > N_0} \frac{1}{(q_{i-B})^{2s}} - \frac{1}{p_i^s p_{i+1}^s} \right|$$

$$(2) \qquad = \left| \sum_{i > N_0} \frac{1}{q_{i-B}^{2s}} - \Big(\frac{1}{q_{i-B}^s} + \frac{1}{p_{i+1}^s} - \frac{1}{q_{i-B}^s} \Big) \Big(\frac{1}{q_{i-B}^s} + \frac{1}{p_i^s} - \frac{1}{q_{i-B}^s} \Big) \right|.$$

By considering the real and imaginary parts of q_j^s, and using the mean value theorem of freshman calculus, $i > j$ implies

$$(3) \qquad \left| \frac{1}{q_i^s} - \frac{1}{q_j^s} \right| \le \frac{(q_i - q_j) 4 \mid s \mid}{q_j^{\sigma+1}}.$$

Thus, multiplying out the RHS of (2),

$$\text{RHS of (2)} = O\left(\sum_{i > N_0} \frac{1}{q_{i-B}^{\sigma}} \left(\frac{q_{i+B+1} - q_{i-B}}{q_{i-B}^{1+\sigma}}\right) \mid s \mid\right)$$

(4)
$$+ O\left(\sum_{i > N_0} \frac{(q_{i+B+1} - q_{i-B})^2}{q_{i-B}^{2+2\sigma}} \mid s \mid^2\right).$$

$\square$

Lemma 4.7.

$$\sum_i \frac{q_{i+k} - q_i}{q_i^{1+\sigma}}$$

converges for $\sigma > 0, k \geq 1$.

Proof. In a 1930's article, Cramér noted that

$$\sum_i \frac{q_{i+1} - q_i}{q_i \log^a q_i}$$

converges for $a > 1$, since the average value of $q_{i+1} - q_i$ is $\log q_i$ [Cra]. So assume $k > 1$, and let $S_x = \Sigma_{q_i \leq x} q_{i+k} - q_i$. Basic facts about primes imply $S_x = O(x)$ (k fixed). Thus, by partial summation,

$$\sum_{q_i \leq x} \frac{q_{i+k} - q_i}{q_i^{1+\sigma}} = O\left(\frac{S_x}{x^{1+\sigma}}\right) + \sum_{q_i \leq x} O(q_i)\left(\frac{1}{q_i^{1+\sigma}} - \frac{1}{q_{i+1}^{1+\sigma}}\right)$$

$$= o(1) + \sum_{q_i \leq x} \frac{O(q_i)(q_{i+1} - q_i)}{q_i^{2+\sigma}} = O(1),$$

using (3) and Cramér's result. $\square$

Continuing with the proof of 4.4, Lemma 4.7 implies the series on the RHS of (4) converge for $\sigma > 0$. The same method can be used to deal with terms in (1) satisfying $\omega(n) \geq 2$ and $\nu(n) = m, 2 < m < N_1$. For a fixed composition α of $m, \alpha_0 + \alpha_1 + \ldots + \alpha_k = m$, the difference between

$$\sum_{i > N_0} \frac{1}{p_i^{\alpha_0 s} p_{i+1}^{\alpha_1 s} \ldots p_{i+k}^{\alpha_k s}} \frac{1}{\alpha_0} \prod_{j=1}^{k} \binom{\alpha_j + \alpha_{j-1} - 1}{\alpha_j}$$

and

$$\sum_{i > N_0} \frac{1}{q_{i-B}^{ms}} \frac{1}{\alpha_0} \prod_{j=1}^{k} \binom{\alpha_j + \alpha_{j-1} - 1}{\alpha_j}$$

is bounded in modulus by a sum of terms of the form

(5) $$\sum_i O(1) \frac{(q_{i+a_0+B+1} - q_{i-B})(q_{i+a_1+B+1} - q_{i-B}) \cdots (q_{i+a_d+B+1} - q_{i-B})}{q_{i-B}^{m\sigma+d+1}},$$

with $1 \le a_0 < a_1 < \cdots < a_d \le k$. Using Lemma 4.7 and elementary estimates, this also converges for $\sigma > 0$. Putting everything together, for σ sufficiently large we now have

$$\zeta_2(s) = \prod_{i=1}^{N_0} C_i(s) \exp\Big(\sum_{\substack{1 \le m \\ N_0 < i}} \frac{1}{mp_i^{ms}} + \sum_{\substack{2 \le m < N_1 \\ N_0 < i}} \frac{d_m}{q_{i-B}^{ms}} + g_1(s)\Big),$$

where $g_1(s)$ is bounded in $\sigma > \delta$ and d_m is a certain positive rational number. This can be rewritten as

$$(6) \qquad \zeta_2(s) = \prod_{i=1}^{N_0} C_i(s) \exp\Big(\sum_{\substack{1 \le m < N_1 \\ N_0 - B \le i}} \frac{h(m)}{q_i^{ms}} + g_2(s)\Big)$$

with $h(m)$ also in $\mathbb{Q}^+$ and g_2 bounded in $\sigma > \delta$. By construction,

$$(7) \qquad h(m) = \sum_{\substack{\alpha_0 + \alpha_1 + \ldots = m \\ \alpha_i \ge 1}} \frac{1}{\alpha_0} \prod_{j \ge 1} \binom{\alpha_j + \alpha_{j-1} - 1}{\alpha_j},$$

and

$$(8) \qquad \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{\cdots}}}} = \exp\Big(\sum_{m=1}^{\infty} x^m h(m)\Big) = \prod_{n \ge 1} \frac{1}{(1 - x^n)^{a_n}},$$

say. Eq. (8) implies $a_n \in \mathbb{Z}$, and by taking logarithms,

$$(9) \qquad a_n = \sum_{d \mid n} \frac{\mu(d)}{d} h(n/d),$$

so $a_1 = h(1) = 1$ and $a_2 = h(2) - h(1)/2 = 1$.

Lemma 4.8. $2^{n-3} \le a_n \le 8^{n+1}$.

Proof. Eq. (9) implies

$$h(n) - \sum_{\substack{d \mid n \\ d < n}} h(d) \le a_n \le h(n) + \sum_{\substack{d \mid n \\ d < n}} h(d).$$

Lemma 4.2 is equivalent to the statement $h(m) \le 8^m$; this implies the upper bound on a_n is $\le 8^n + \ldots + 8 + 1 < 8^{n+1}$. For the lower bound, given a composition α of d, $\alpha_0 + \alpha_1 + \ldots + \alpha_k = d$, where $d \mid n$ and $d \le n/2$, let $\beta(\alpha)$ be the composition $\alpha_0 + \alpha_1 + \ldots + \alpha_k + u + 1 + 1 + \ldots + 1$ of n, where u is the greatest integer in $(n+1)/2$ and $\beta(\alpha)$ ends in $n - d - u$ ones. Clearly the product corresponding to $\beta(\alpha)$ in (7) is at least as big as the product corresponding to α; hence

$$a_n \geq \sum_{\substack{\alpha_0+\alpha_1+\ldots+\alpha_j=n \\ \alpha \text{ doesn't end in } u1^e \\ \text{for some } e \geq 0}} \frac{1}{\alpha_0} \prod_{i=1}^{j} \binom{\alpha_i + \alpha_{i-1} - 1}{\alpha_i}$$

$$\geq \sum_{\substack{\alpha_0+\alpha_1+\ldots+\alpha_j=n \\ \alpha \text{ doesn't end in } u1^e \\ \text{for some } e \geq 0 \text{ and } \alpha_0 = 1}} 1$$

$$= \sum_{\substack{\alpha_0+\alpha_1+\ldots=n \\ \alpha_0=1}} 1 - \sum_{e \geq 0} \left\{ \begin{array}{l} \# \text{ of compositions of } n, \\ \alpha_0 = 1, \alpha \text{ ends in } u1^e \end{array} \right\}$$

$$= 2^{n-2} - \sum_{e=0}^{n-u-2} 2^{n-u-2-e} - 1 \geq 2^{n-3} \text{ for } n \geq 3. \qquad \square$$

By the argument below (1), if we replace $g_2(s)$ in (6) by

$$g_3(s) + \sum_{\substack{N_0-B \leq i \\ N_1 \leq m}} \frac{h(m)}{q_i^{ms}}$$

then g_3 is also bounded in $\sigma > \delta$. We now have

$$\zeta_2(s) = \exp(g_3(s)) \prod_{i=1}^{N_0} C_i(s) \prod_{i \geq N_0-B} \prod_{k=1}^{\infty} \frac{1}{(1 - q_i^{-ks})^{a_k}}.$$

Since $a_k \in \mathbb{Z}^+$,

$$\left| \prod_{i \geq N_0-B} \prod_{k=N_1+1}^{\infty} \frac{1}{(1 - q_i^{-ks})^{a_k}} \right| \leq \exp\left(\sum_{\substack{N_0-B \leq i \\ N_1 \leq m}} \frac{h(m)}{q_i^{m\sigma}} \right)$$

$$= \exp(g_4(s) \text{ say, and}$$

$$\prod_{i=1}^{N_0-B} \prod_{k=1}^{N_1} \frac{1}{(1 - q_i^{-ks})^{a_k}} = \exp(g_5(s)),$$

where g_4 and g_5 are bounded in $\sigma > \delta$. Finally we get

$$\zeta_2(s) = \exp(g(s)) \prod_{i=1}^{N_0} C_i(s) \prod_{i=1}^{\infty} \prod_{k=1}^{N_1} \frac{1}{(1 - q_i^{-ks})^{a_k}}$$

for σ sufficiently large, where $|g(s)| \leq g_3 + g_4 + g_5$ is bounded in $\sigma > \delta$. Theorem 4.4 now follows from the analytic continuation of the zeta function. $\square$

Remarks. The proof of Theorem 4.4 can easily be modified to show that given $\delta > 0$, there exist integers N_r, M_r such that $N_r, M_r \to \infty$ as $\delta \to 0$ and

$$\zeta_r(s) = \exp(g(s, r)) \prod_{i=1}^{N_r} C_{i,r}(s) \prod_{k=1}^{M_r} \zeta(ks)^{a_k(r)}, \qquad r > 1,$$

where $a_k(r) \in \mathbb{Z}^+, a_1(r) \equiv 1, a_2(r) = r - 1$, and $2^{k-3} \leq a_k(r) \leq 2^{rk}r^k$ for $k \geq 3$. Here $C_{i,r}(s)$ is the ith branching continued fraction term in the product formula for $\zeta_r(s)$ (section 2), and $g(s, r)$ is bounded in $\sigma > \delta$.

For "unbounded" permutations, the method of Theorem 4.4 can at least be used to continue $\zeta_2(s)$ to the line $\sigma = 1/2$. For example, the series

$$\sum_i \frac{1}{p_i^s p_{i+1}^s}$$

is bounded in modulus by $2\sum_p 1/p^{2\sigma}$, and terms where $\nu(n) > 2$ also converge for $\sigma > 1/2$. But the procedure breaks down when trying to continue $\zeta_2(s)$ beyond the half-line. Say $p_{2i} = q_i$, and $p_{2i+1} > 10^i$; then the difference between

$$\sum_i \frac{1}{p_i^s p_{i+1}^s} \quad \text{and} \quad \sum_i \frac{1}{q_i^{2s}}$$

only converges for $\sigma > 1/2$ since

$$\sum_i \frac{1}{p_i^\sigma p_{i+1}^\sigma} < 2\sum_i \frac{1}{10^{i\sigma}} < \infty \qquad \text{for } \sigma > 0.$$

5. The partial sums of $\mu_r(n)$

Recall that $M(x) = \Sigma_{n \le x}\mu(n)$, and that the RH is equivalent to the condition $M(x) = O(x^{1/2+\epsilon})$ for every $\epsilon > 0$. The RH also implies a similar condition for $r > 1$:

Theorem 5.1. *Assume the RH. Then*

$$\sum_{n \le x} \mu_r(n) = O(x^{1/2+\epsilon}) \qquad \forall \epsilon > 0.$$

Proof. Theorem 5.1 is a special case of the following more general result: □

Lemma 5.2. *Start with an arbitrary ordering of the primes $p_1, p_2, \ldots$, and let $\mathcal{D}_r$ denote the following infinite set:*

$$\mathcal{D}_r = \left\{ p_i p_j \mid 1 \le i < j \le i + r - 1 \right\}.$$

Let $\mathcal{D}$ be any (possibly infinite) subset of $\mathcal{D}_r$. Then, assuming the RH,

$$\sum_{\substack{n \le x \\ d|n \implies d \notin \mathcal{D}}} \mu(n) = O(x^{1/2+\epsilon}) \text{ for every } \epsilon > 0.$$

Proof of Lemma 5.2. By inclusion-exclusion,

$$\sum_{\substack{n \le x \\ d|n \implies d \notin \mathcal{D}}} \mu(n) = M(x) - \sum_{d \in \mathcal{D}} \mu(d) M(x/d, d) + \sum_{\substack{d_1, d_2 \in \mathcal{D} \\ d_1 < d_2}} \mu([d_1 d_2]) M\left(\frac{x}{[d_1 d_2]}, [d_1 d_2]\right)$$

$$(10) \qquad - \sum_{\substack{d_1, d_2, d_3 \in \mathcal{D} \\ d_1 < d_2 < d_3}} \mu([d_1 d_2 d_3]) M\left(\frac{x}{[d_1 d_2 d_3]}, [d_1 d_2 d_3]\right) + - + \ldots$$

where $[d_1 d_2 \ldots d_s] = $ the least common multiple of $d_1, d_2, \ldots, d_s$ and

$$M(x, d) = \sum_{\substack{n \le x \\ (n,d)=1}} \mu(n).$$

It turns out that the RH also implies $M(x,d) = O(x^{1/2+\epsilon})$ for any $d \leq x$; first we show by induction that

$$M(x,d) = \sum_{\substack{m \\ p|m \implies p|d}} M(x/m). \quad \square$$

Proof. The proof is actually an induction on ordered pairs (j,x), where $j = \omega(d)$. Clearly we can assume d is squarefree. For $j = 0, M(x,1) = M(x)$. The case $x = 0$ reduces to $0 = 0$. For $j > 1$, by inclusion-exclusion,

$$M(x,d) = M(x) - \sum_{p|d} \mu(p) M(x/p, p) + \sum_{\substack{p_i, p_j | d \\ i < j}} \mu(p_i p_j) M(x/p_i p_j, p_i p_j) - + - \dots$$

$$= \sum_{q|d} M(x/q, q) = \sum_{q|d} \sum_{\substack{k \\ p|k \implies p|q}} M(x/qk) \qquad \text{(by induction)}$$

$$= \sum_{\substack{k \\ p|k \implies p|d}} M(x/k). \quad \square$$

Next note that for any y,

$$|M(y,d)| \leq \sum_{\substack{k \\ p|k \implies p|d}} |M(y/k)|$$

$$= O\left(\sum_{\substack{k \\ p|k \implies p|d}} (y/k)^{1/2+\epsilon} \right) \qquad \text{by the RH}$$

(11) $$= O\left(y^{1/2+\epsilon} \sum_{\substack{k \\ p|k \implies p|d}} 1/k^{1/2+\epsilon} \right) = O\left(y^{1/2+\epsilon} \prod_{p|d} \frac{1}{1 - p^{-1/2-\epsilon}} \right).$$

In (10) we can assume $d \leq x$, in which case

$$\prod_{p|d} \frac{1}{1 - p^{-1/2-\epsilon}} \leq \prod_{i=1}^{N} \frac{1}{1 - q_i^{-1/2-\epsilon}},$$

where q_i is the ith largest prime, and N is chosen so that $\prod_{i=1}^{N} q_i \leq x < \prod_{i=1}^{N+1} q_i$. Thus $\sum_{i=1}^{N} \log q_i \leq \log x$. Since $\sum_{i=1}^{N} \log q_i = O(q_N)$ by the Prime Number Theorem, $q_N = O(\log x)$ and thus

$$\prod_{i=1}^{N} \frac{1}{1 - q_i^{-1/2-\epsilon}} = O\left(\exp O\left(\sum_{p \leq \log x} 1/p^{1/2+\epsilon} \right) \right) = O(\exp(O(\log^{1/2} x))).$$

Using this estimate in (11), we get $M(y,d) = O(y^{1/2+\epsilon} \exp(O(\log^{1/2} x))) = O(y^{1/2+\epsilon} x^{\epsilon})$ uniformly in d for $d \leq x$. Plugging this into (10) results in

$$\left| \sum_{\substack{n \leq x \\ d|n \implies d \notin \mathcal{D}}} \mu(n) \right| \leq |M(x)| + \sum_{d \in \mathcal{D}} |M(x/d, d)| + \sum_{\substack{d_1, d_2 \in \mathcal{D} \\ d_1 < d_2}} \left| M(\frac{x}{[d_1 d_2]}, [d_1 d_2]) \right| + \dots$$

$$= O(x^{\epsilon}) O(x^{1/2+\epsilon}) + O(x^{\epsilon}) \sum_{d \in \mathcal{D}} O((x/d)^{1/2+\epsilon})$$

$$+ O(x^{\epsilon}) \sum_{\substack{d_1, d_2 \in \mathcal{D} \\ d_1 < d_2}} O((x/[d_1 d_2])^{1/2+\epsilon}) + \dots$$

(where the implied constants are uniformly bounded)

$$(12) \qquad\qquad = O(x^{1/2+2\epsilon} \sum_{k \leq x} \frac{c_k}{k^{1/2+\epsilon}})$$

where c_k is the number of ways that k can be written in the form $[d_1 d_2 \cdots d_j], d_i \in \mathcal{D}$.

Given k with $c_k > 0$, we can uniquely decompose k into squarefree "blocks" of primes $B_1 B_2 \dots B_m$ where

a) all the primes occurring in block B_i occur in the list $p_1 p_2 p_3 \dots$ before any of the primes in block B_{i+1} for $1 \leq i \leq m-1$,

b) if p_j divides B_i and p_k divides B_{i+1} then $k - j \geq r$ (i.e. there are at least $r-1$ primes, not dividing k, between blocks B_i and B_{i+1}), and

c) the number of blocks is maximal.

Example. If $k = p_2 p_4 p_5 p_8 p_{10} p_{13} p_{14}$, and $r = 3$, then $B_1 = p_2 p_4 p_5$, $B_2 = p_8 p_{10}$, and $B_3 = p_{13} p_{14}$.

Now if B_j consists of the product of an even number of primes, say $B_j = p_{t_1} p_{t_2} \cdots p_{t_{2q}}$, decompose B_j into "mini-blocks" as follows:

$$B_j = (p_{t_1} p_{t_2})(p_{t_3} p_{t_4}) \cdots (p_{t_{2q-1}} p_{t_{2q}}).$$

If B_j contains an odd number of primes, which must be greater than 1 if $c_k > 0$, say $B_j = p_{t_1} \cdots p_{t_{2q+1}}$, decompose B_j into "mini-blocks" as follows:

$$B_j = (p_{t_1} p_{t_2}) \cdots (p_{t_{2q-3}} p_{t_{2q-2}})(p_{t_{2q-1}} p_{t_{2q}} p_{t_{2q+1}}).$$

Hence

$$\sum_{\substack{k \leq x \\ c_k > 0}} \frac{1}{k} \leq \prod_{p_i \leq x} (1 + \frac{1}{p_i p_{i+1}} + \frac{1}{p_i p_{i+2}} + \dots + \frac{1}{p_i p_{i+r-1}}$$

$$+ \frac{1}{p_i p_{i+1} p_{i+2}} + \frac{1}{p_i p_{i+1} p_{i+3}} + \dots + \frac{1}{p_i p_{i+r-1} p_{i+2r-2}}),$$

since the terms in the product on the right are the only possible mini-blocks in which p_i is the prime with the smallest subscript.

Now a given prime p_i occurs in terms in $\mathcal{D}$ of the form $p_i p_{i+j}$ or $p_{i-j} p_i$ for $1 \leq j \leq r-1$, so divides at most $2r-2$ elements in $\mathcal{D}$. Thus there are at most $\omega(k)(2r-2)$ elements in $\mathcal{D}$ which divide k. There are at most $2^{\omega(k)(2r-2)}$ subsets

TABLE I. The values of $M_r(x)$

X	r	$M_r(X)$	Max $/x^{1/2}$	X	r	$M_r(X)$	Max $/x^{1/2}$
10^5	1	-48	0.417	2×10^7	1	-953	0.324
	2	307	1.236		2	-5958	1.343
	3	1349	4.320		3	-17030	3.818
	4	2724	8.620		4	-16134	3.616
10^6	1	212	0.368	4×10^7	1	-333	0.407
	2	178	0.510		2	-8954	1.507
	3	2002	2.167		3	-31911	5.090
	4	6623	6.679		4	-41805	6.636
10^7	1	1037	0.361	6.5×10^7	1	1904	0.353
	2	-2276	1.127		2	-10078	1.410
	3	-7018	2.290		3	-45474	5.729
	4	-1599	2.243		4	-69833	8.698

of these, which gives an upper bound for c_k. Hence

$$\sum_{k \leq x} \frac{c_k}{k^{1/2+\epsilon}} \leq \prod_{p_i \leq x} (1 + \frac{4^{2r-2}}{(p_i p_{i+1})^{1/2+\epsilon}} + \frac{4^{2r-2}}{(p_i p_{i+2})^{1/2+\epsilon}} + \ldots + \frac{4^{2r-2}}{(p_i p_{i+r-1})^{1/2+\epsilon}}$$

$$+ \frac{8^{2r-2}}{(p_i p_{i+1} p_{i+2})^{1/2+\epsilon}} + \frac{8^{2r-2}}{(p_i p_{i+1} p_{i+3})^{1/2+\epsilon}} + \ldots + \frac{8^{2r-2}}{(p_i p_{i+r-1} p_{i+2r-2})^{1/2+\epsilon}})$$

$$\leq \prod_{p_i \leq x} (1 + \frac{4^{2r-2}(r-1)}{\min(p_i, p_{i+1}, \ldots, p_{i+r-1})^{1+2\epsilon}} + \frac{8^{2r-2}\binom{2r-2}{2}}{\min(p_i, p_{i+1}, \ldots, p_{i+2r-2})^{3/2+3\epsilon}})$$

(since there are $\binom{2r-2}{2}$ triples $p_i p_j p_t, 1 \leq i < j < t \leq i + 2r - 2, i$ fixed)

$$\leq \prod_{p_i \leq x} (1 + \frac{4^{2r-2}(r-1)}{p_i^{1+2\epsilon}} + \frac{8^{2r-2}\binom{2r-2}{2}}{p_i^{3/2+3\epsilon}})^{2r-1}$$

(note the exponent $2r-1$; $p_i = \min(p_a, p_{a+1}, \ldots, p_{a+2r-2})$ for at most $2r-1$ values of a)

$$\leq \prod_{p_i \leq x} (1 + \frac{d_r}{p_i^{1+2\epsilon}})^{2r-1} \leq A_r \prod_{2d_r < p_i \leq x} (1 + \frac{d_r}{p_i^{1+2\epsilon}})^{2r-1}$$

for some constants d_r and A_r. Now

$$\log(\prod_{2d_r < p_i \leq x} (1 + \frac{d_r}{p_i^{1+2\epsilon}})^{2r-1}) = O(\sum_{2d_r < p_i \leq x} \frac{d_r(2r-1)}{p_i^{1+2\epsilon}}) = O(1).$$

Thus

$$\sum_{k \leq x} \frac{c_k}{k^{1/2+\epsilon}} = O(1),$$

which together with (12) completes the proof of Lemma 5.2. $\qquad\square$

In Table I, $M_r(x) = \sum_{n \leq x} \mu_r(n)$, and $\mathrm{Max}/x^{1/2} = (\max_{y \leq x} |M_r(y)|)/(x^{1/2})$. The author would like to thank Charles Grassl of CRAY Research for running the program on a CRAY C-90 at Mendota Heights, Minnesota. The CPU time was 86 seconds.

6. Some Identities in the Ring of Formal Power Series

Several of the combinatorial identities involving $\zeta_r(s)$ have more general forms in the ring of formal power series, discovered while working on problems involving *rook theory*. An $n \times n$ *chessboard* is an $n \times n$ grid of squares; the lower left hand square has (column,row) coordinates $(1,1)$. A *Ferrers board B* is a subset of squares of the chessboard with the property that if square $(i,j) \in B$ then all of the squares below it and to the right of square (i,j) are also in B. A Ferrers board can be described in terms of its "column heights", $c_1 \leq c_2 \leq \ldots \leq c_n$, where B consists of the lowest c_i squares in the ith column, for $1 \leq i \leq n$. See Figure 1. Note that the first several columns of this board have height 0.

One of the most interesting theorems about Ferrers boards is due to Goldman, Joichi, and White, which says [GJW];

$$(13) \qquad \sum_{k=0}^{n} x(x-1)\cdots(x-k+1)r_{n-k}(B) = \prod_{i=1}^{n}(x + c_i - i + 1), \qquad x \in \mathbb{C},$$

where $r_j(B) =$ the number of ways of placing j non-attacking rooks (no two in the same row or column) on B. Using this, the author showed in his dissertation [Ha1] that rook placements are related to *compositions* of vectors. Given a vector $\mathbf{v} \in \mathbb{N}^t$, let $g_k(\mathbf{v})$ be the number of ways of writing $\mathbf{v}$ as the sum of k nonzero vectors, each in $\mathbb{N}^t$, and with each coordinate equal to zero or one. These sums were originally studied by MacMahon; they are called *unitary compositions* ([Mac], p.158).
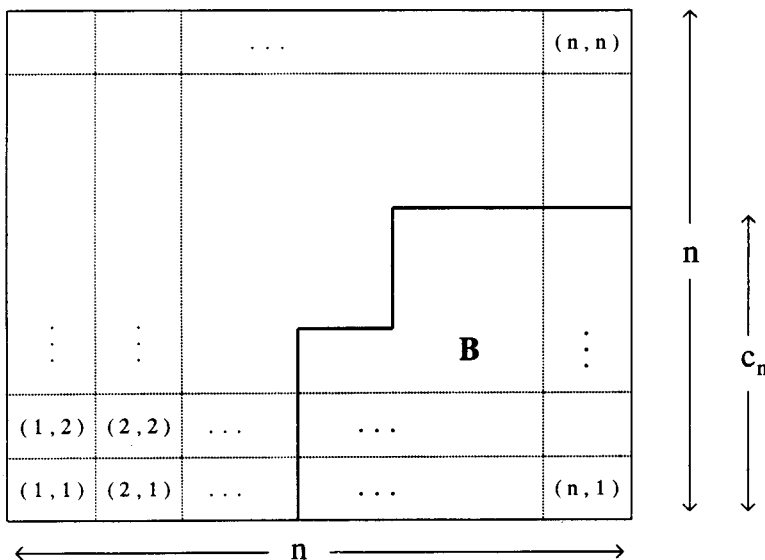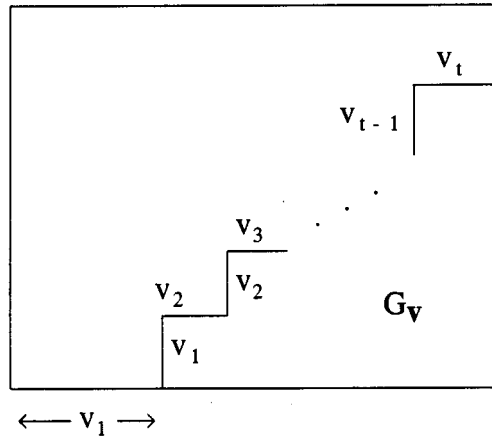

FIGURE 1. The Ferrers board B

FIGURE 2. The Ferrers board $G_{\mathbf{v}}$

Example. Let $\mathbf{v} = (2,1)$. Then $g_2(\mathbf{v}) = 2$ since

$$(2,1) = (1,1) + (1,0) = (1,0) + (1,1).$$

Theorem 3.2.6 of [Ha1] says that

$$g_k(\mathbf{v}) = \frac{k! \, r_{n-k}(G_{\mathbf{v}})}{\prod_i v_i!},$$

where $n = \sum v_i$ and $G_{\mathbf{v}}$ is the Ferrers board in Figure 2 (the first v_1 columns have height zero).

The boards $G_{\mathbf{v}}$ are also connected to Simon Newcomb's problem (originally studied by MacMahon) which asks for the number of permutations of a multiset with a specified number of descents. A permutation σ of a multiset M is a linear list of the elements of $M : \sigma_1 \sigma_2 \cdots \sigma_{\#M}$. A *descent* of σ is a value of i such that $\sigma_i > \sigma_{i+1}$. Let $N_k(\mathbf{v})$ be the number of permutations of the multiset consisting of v_i copies of $i, 1 \leq i \leq t$, with exactly $k-1$ descents. Riordan and Kaplansky showed that [K-R]

$$N_k(\mathbf{v}) = \frac{T_{k-1}(G_{\mathbf{v}})}{\prod_i v_i!}$$

where $T_j(B)$ equals the number of ways of placing j non-attacking rooks on the $n \times n$ chessboard with exactly j on the board B (which is a subset of the $n \times n$ grid). For more recent work on compositions and Simon Newcomb's problem see [An1], [An2], [An3], [An4], [D-R], [F-S], [Ha2], [Si1], and [Si2].

There is a natural way of generalizing the Simon Newcomb problem which has been studied by Rawlings [Raw]. Define an r-descent of a permutation σ to be a value of i such that $\sigma_i > \sigma_{i+1} + r - 1$. Let $N_k(\mathbf{v}, r)$ be the number of permutations (of the multiset consisting of v_i copies of $i, 1 \leq i \leq t$) with exactly $k - 1$ r-descents (actually Rawlings considered a q-version which depends on both r and q). Another result from [Ha1] is that

$$(14) \qquad\qquad N_k(\mathbf{v}, r) = \frac{T_{k-1}(G_{\mathbf{v},r})}{\prod_i v_i!},$$
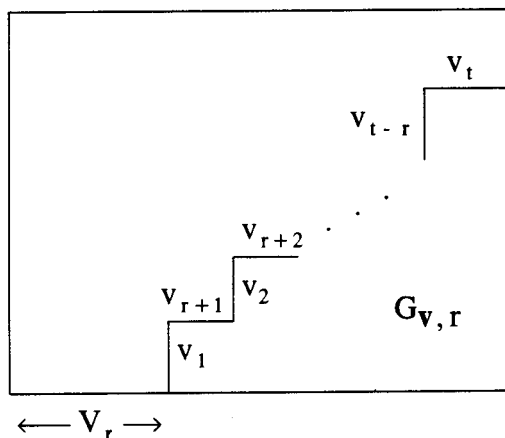
FIGURE 3. The Ferrers board $G_{\mathbf{v},r}$

where $G_{\mathbf{v},r}$ is the Ferrers board of Figure 3 (here $V_r = v_1 + v_2 + \ldots + v_r$).

Can the rook numbers $r_j(G_{\mathbf{v},r})$ be interpreted in terms of compositions as in the $r = 1$ case? Yes. Given a vector $\mathbf{v} = (v_1, v_2, \ldots, v_t)$, call $\mathbf{v}$ *r-unitary* if

a) $v_i = 0$ or 1 for all i, and

b) any two ones in $\mathbf{v}$ are separated by at least $r - 1$ zeros (i.e. if $v_i = v_j = 1$ and $i < j$, then $j - i \geq r$).

For example, the vector $(1,0,1,0,0,1,0)$ is 2-unitary but not 3-unitary.

Let $g_k(\mathbf{v}, r)$ be the number of ways of writing $\mathbf{v}$ as the sum of k nonzero r-unitary vectors. The following result is derived in [Ha2].

Theorem 6.1.

$$g_k(\mathbf{v}, r) = \frac{k! \, r_{n-k}(G_{\mathbf{v},r})}{\prod_i v_i!}.$$

The definition of $g_k(\mathbf{v}, r)$ can be expressed in terms of generating functions as follows. Let $E_r(x_1, x_2, \ldots, x_t)$ denote the following function in the ring of formal power series:

$$E_r(x_1, x_2, \ldots, x_t) = \sum_{\substack{\mathbf{v} \in \mathbb{N}^t \\ \mathbf{v} \text{ is } r\text{-unitary}}} \prod_i x_i^{v_i},$$

and let

$$E_r(x_1, x_2, \ldots) = \lim_{t \to \infty} E_r(x_1, x_2, \ldots, x_t).$$

Then by definition,

$$\sum_{\mathbf{v} \in \mathbb{N}^t} \prod_i x_i^{v_i} g_k(\mathbf{v}, r) = (E_r(x_1, x_2, \ldots, x_t) - 1)^k.$$

In [Ha2], (15) and the next theorem are shown to follow from Theorem 6.1 and (13). A self-contained proof which doesn't rely on rook theory is included here.

Theorem 6.2. *In the ring of formal power series, for any $x \in \mathbb{C}$,*

$$E_r(x_1, x_2, \ldots)^x = \sum_{\mathbf{v}} \prod_i x_i^{v_i} \binom{x - v_{i-1} - v_{i-2} - \ldots - v_{i-r+1}}{v_i}$$

(for $r = 1$ this reduces to the binomial theorem since $E_1(x_1, x_2, \ldots) = \prod_i (1 + x_i)$).

Proof. We start by showing that

$$(15) \qquad F_r(x_1, x_2, \ldots, x_t)^x = \sum_{v \in \mathbb{N}^t} \prod_i x_i^{v_i} \binom{x - 1 + v_i + v_{i-1} + \ldots + v_{i-r+1}}{v_i},$$

where

$$F_r(x_1, x_2, \ldots, x_t) = \sum_{\mathbf{v} \in \mathbb{N}^t} \prod_i x_i^{v_i} \binom{v_i + v_{i-1} + \ldots + v_{i-r+1}}{v_i}.$$

First assume that $x \in \mathbb{N}$, and proceed by induction on x. For $x = 1$, (15) is trivial. For $x > 1$,

$$F_r(x_1, x_2, \ldots, x_t)^x = F_r(x_1, x_2, \ldots, x_t)^{x-1} F_r(x_1, x_2, \ldots, x_t)$$

$$= \left(\sum_{\mathbf{v} \in \mathbb{N}^t} \prod_i x_i^{v_i} \binom{x - 2 + v_i + v_{i-1} + \ldots + v_{i-r+1}}{v_i} \right)$$

$$\times \left(\sum_{\mathbf{v} \in \mathbb{N}^t} \prod_i x_i^{v_i} \binom{v_i + v_{i-1} \ldots + v_{i-r+1}}{v_i} \right)$$

(by the induction hypothesis)

$$= \sum_{\mathbf{v} \in \mathbb{N}^t} \prod_i x_i^{v_i} \sum_{\mathbf{0} \subseteq \mathbf{w} \subseteq \mathbf{v}} \prod_i \binom{x - 2 + v_i - w_i + \ldots + v_{i-r+1} - w_{i-r+1}}{v_i - w_i}$$

$$\times \binom{w_i + \ldots + w_{i-r+1}}{w_i}.$$

Now we need to do a sub-induction on t. For $t = 1$,

$$\sum_{w_1 = 0}^{v_1} \binom{x - 2 + v_1 - w_1}{v_1 - w_1} \binom{w_1}{w_1}$$

$$= \text{COEFFICIENT OF } (t^{v_1}) \text{ IN } \frac{1}{(1-t)^{x-1}} \frac{1}{1-t} = \binom{x - 1 + v_1}{v_1}.$$

For $t > 1$, with $\mathbf{v}' = (v_1, \ldots, v_{t-1})$,

$$\sum_{\mathbf{0} \subseteq \mathbf{w} \subseteq \mathbf{v}} \prod_i \binom{x - 2 + v_i - w_i + v_{i-1} - w_{i-1} + \ldots + v_{i-r+1} - w_{i-r+1}}{v_i - w_i}$$

$$\times \binom{w_i + \ldots + w_{i-r+1}}{w_i}$$

$$= \sum_{\mathbf{0}^{t-1} \subseteq \mathbf{w} \subseteq \mathbf{v}'} \prod_{i=1}^{t-1} \binom{x - 2 + v_i - w_i + \ldots + v_{i-r+1} - w_{i-r+1}}{v_i - w_i}$$

$$\times \binom{w_i + \ldots + w_{i-r+1}}{w_i} \sum_{w_t=0}^{v_t} \binom{x - 2 + v_t - w_t + \ldots + v_{t-r+1} - w_{t-r+1}}{v_t - w_t}$$

(16)
$$\times \binom{w_t + \ldots + w_{t-r+1}}{w_t}.$$

The sum over w_t is the COEFFICIENT OF (t^{v_t}) IN

$$\frac{1}{(1-t)^{x-1+v_{t-1}-w_{t-1}+\ldots+v_{t-r+1}-w_{t-r+1}}(1-t)^{w_{t-1}+\ldots+w_{t-r+1}+1}}$$

$$= \text{COEFFICIENT OF } (t^{v_t}) \text{ IN } \frac{1}{(1-t)^{x+v_{t-1}+\ldots+v_{t-r+1}}}$$

$$= \binom{x - 1 + v_t + \ldots + v_{t-r+1}}{v_t}.$$

Plugging this into (16) completes the induction on t and x. Next we need to show (15) is true for $x \in \mathbb{C}$. Setting $(F_r(x_1, x_2, \ldots, x_t) - 1)^k = \sum_{\mathbf{v}} \prod_i x_i^{v_i} f_k(\mathbf{v}, r)$, we get

$$F_r(x_1, x_2, \ldots, x_t)^x = (1 + F_r(x_1, x_2, \ldots, x_t) - 1)^x$$

$$= \sum_{k=0}^{\infty} \binom{x}{k} (F_r(x_1, x_2, \ldots, x_t) - 1)^k$$

$$= \sum_{\mathbf{v} \in \mathbb{N}^t} \prod_i x_i^{v_i} \sum_{k=0}^{\infty} \binom{x}{k} f_k(\mathbf{v}, r)$$

$$= \sum_{\mathbf{v} \in \mathbb{N}^t} \prod_i x_i^{v_i} \binom{x - 1 + v_i + v_{i-1} + \ldots + v_{i-r+1}}{v_i} \text{ by (15).}$$

Comparing the coefficient of $\prod_i x_i^{v_i}$ on both sides, we see that

(17)
$$\sum_{k=0}^{\infty} \binom{x}{k} f_k(\mathbf{v}, r) = \prod_{i=1}^{t} \binom{x - 1 + v_i + v_{i-1} + \ldots + v_{i-r+1}}{v_i}.$$

From the definition of $f_k(\mathbf{v}, r)$ given a few lines above, $f_k(\mathbf{v}, r) = 0$ if $k > \sum_i v_i$. Both sides of (17) are thus polynomials in x which agree if $x \in \mathbb{N}$ and hence are identically equal. To show that Theorem 6.2 follows, first let $x = -1$ in (15);

$$F_r(x_1, x_2, \ldots, x_t)^{-1} = \sum_{\mathbf{v} \in \mathbb{N}^t} \prod_i x_i^{v_i} \binom{-2 + v_i + v_{i-1} + \ldots + v_{i-r+1}}{v_i}.$$

It is not hard to see that

$$\prod_{i=1}^{t} \binom{-2 + v_i + v_{i-1} + \ldots + v_{i-r+1}}{v_i} = \begin{cases} 0, & \text{if } \mathbf{v} \text{ is not } r\text{-unitary;} \\ (-1)^{v_1 + \ldots + v_t}, & \text{if } \mathbf{v} \text{ is } r\text{-unitary} \end{cases}$$

which implies

(18)
$$F_r(-x_1, -x_2, \ldots, -x_t)^{-1} = E_r(x_1, x_2, \ldots, x_t),$$

and letting $t \to \infty$ we get

$$E_r(x_1, x_2, \dots)^x = F_r(-x_1, -x_2, \dots)^{-x}$$

$$= \sum_{\mathbf{v}} \prod_i (-x_i)^{v_i} \binom{-x - 1 + v_i + v_{i-1} + \dots + v_{i-r+1}}{v_i}.$$

$$= \sum_{\mathbf{v}} \prod_i x_i^{v_i} \binom{x - v_{i-1} - v_{i-2} - \dots - v_{i-r+1}}{v_i}$$

since

$$\binom{m}{k} = (-1)^k \binom{-m + k - 1}{k}. \qquad \square$$

Remark. The coefficients which occur in $F_r(x_1, x_2, \dots)$ have a combinatorial interpretation:

$$\prod_i \binom{v_i + v_{i-1} + \dots + v_{i-r+1}}{v_i}$$

is the number of permutations (of the multiset having v_i copies of i) with no r-descents.

Continued fraction expansions. Let $CF_r(x_1, x_2, \dots)$ denote the following branching continued fraction, where each term branches into $r - 1$ new terms;

$$\cfrac{1}{1 + \cfrac{x_1}{(1 + \cfrac{x_2}{(1 + \frac{x_3}{(1 + \dots)}) \dots (1 + \frac{x_{r+1}}{(1 + \dots)})}) \dots (1 + \cfrac{x_r}{(1 + \frac{x_{r+1}}{(1 + \dots)}) \dots (1 + \frac{x_{2r-1}}{(1 + \dots)})})}}.$$

These structures have been the subject of several research articles and books, all of which are in Russian or Ukranian and difficult to obtain (see [RNT] Vol. 1a, pp.156-158). The rest of this section consists of several interesting combinatorial properties of these functions, which were derived independently by the author but most likely occur somewhere in the literature.

Theorem 6.3 (For r=2 this result appears in [Per]). *Let $CF_r(x_1, x_2, \dots, x_n)$ denote the finite continued fraction $CF_r(x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$. Then*

$$CF_r(x_1, x_2, \dots, x_n) = \frac{E_r(x_2, x_3, \dots, x_n)}{E_r(x_1, x_2, \dots, x_n)}.$$

Proof. By induction on n, both sides equaling $1/(1 + x_1)$ for $n = 1$. For $n > 1$,

$$CF_r(x_1, x_2, \dots, x_n)$$

$$= \frac{1}{1 + x_1 CF_r(x_2, x_3, \dots, x_n) CF_r(x_3, x_4, \dots, x_n) \cdots CF_r(x_r, x_{r+1}, \dots, x_n)}$$

$$= \frac{1}{1 + x_1 \frac{E_r(x_3, x_4, \dots, x_n)}{E_r(x_2, x_3, \dots, x_n)} \cdots \frac{E_r(x_{r+1}, x_{r+2}, \dots, x_n)}{E_r(x_r, x_{r+1}, \dots, x_n)}} \qquad \text{by induction}$$

$$= \frac{E_r(x_2, \dots, x_n)}{E_r(x_2, x_3, \dots, x_n) + x_1 E_r(x_{r+1}, x_{r+2}, \dots, x_n)}$$

$$= \frac{E_r(x_2, x_3, \dots, x_n)}{E_r(x_1, x_2, \dots, x_n)}. \qquad \square$$

Corollary 6.4.

$$F_r(x_1, x_2, \dots) = \prod_{i=1}^{\infty} CF_r(-x_i, -x_{i+1}, \dots)$$

provided that $E_r(\mid x_1 \mid, \mid x_2 \mid, \dots) < 2$.

Proof. By letting $n \to \infty$ in Theorem 6.3, the product on the right of Corollary 6.4 telescopes and so equals $E_r(-x_1, -x_2, \dots)^{-1} = F_r(x_1, x_2, \dots)$. The condition $E_r(\mid x_1 \mid, \mid x_2 \mid, \dots) < 2$ guarantees that none of the terms in the product have zero for a denominator. $\square$

Theorem 6.5. (For $r = 2$ this result occurs in some unpublished work of Doug Bowman) *In the ring of formal power series,*

$$CF_r(-x_1, -x_2, \dots)^x = \sum_{\mathbf{v}} \prod_i x_i^{v_i} \binom{x + v_1 - 1}{v_1} \prod_{i \geq 2} \binom{v_i + \dots + v_{i-r+1} - 1}{v_i}.$$

Proof. We have

$$CF_r(-x_1, -x_2, \dots, -x_t)^x = \frac{E_r(-x_2, -x_3, \dots - x_t)^x}{E_r(-x_1, -x_2, \dots, -x_t)^x} \qquad \text{(Theorem 6.3)}$$

$$= \sum_{\substack{\mathbf{v} \in \mathbb{N}^t \\ v_1 = 0}} \prod_{i \geq 2} (-x_i)^{v_i} \binom{x - v_{i-1} - \dots - v_{i-r+1}}{v_i}$$

$$\times \sum_{\mathbf{v} \in \mathbb{N}^t} \prod_{i \geq 1} (-x_i)^{v_i} \binom{-x - v_{i-1} - \dots - v_{i-r+1}}{v_i}$$

$$= \sum_{\mathbf{v} \in \mathbb{N}^t} \prod_i (-x_i)^{v_i} \binom{-x}{v_1} \sum_{w_2=0}^{v_2} \binom{x}{w_2} \binom{-x - v_1}{v_2 - w_2}$$

$$\times \sum_{w_3=0}^{v_3} \cdots \sum_{w_t=0}^{v_t} \binom{x - w_{t-1} \dots - w_{t-r+1}}{w_t}$$

$$\times \binom{-x - (v_{t-1} - w_{t-1}) - \dots - (v_{t-r+1} - w_{t-r+1})}{v_t - w_t}.$$

By the Vandermonde convolution, the sum over w_t equals

$$\binom{-v_{t-1} - v_{t-2} - \dots - v_{t-r+1}}{v_t},$$

and proceeding with an inductive argument we end up with

$$CF_r(-x_1, -x_2, \dots, -x_t)^x = \sum_{\mathbf{v} \in \mathbb{N}^t} \prod_i (-x_i)^{v_i} \binom{-x}{v_1} \prod_{i \geq 2} \binom{-v_{i-1} - \dots - v_{i-r+1}}{v_i}$$

$$= \sum_{\mathbf{v} \in \mathbb{N}^t} \prod_i x_i^{v_i} \binom{x + v_1 - 1}{v_1} \prod_{i \geq 2} \binom{v_i + v_{i-1} + \dots + v_{i-r+1} - 1}{v_i}$$

(since $\binom{n}{k} = (-1)^k \binom{-n+k-1}{k}$). Letting $t \to \infty$ finishes the proof. $\square$

Remark. A similar argument shows that

$$\prod_{i=1}^{m} CF_r(-x_i, -x_{i+1}, \dots)^x = \sum_{\mathbf{v}} \prod_i x_i^{v_i} \prod_{i=1}^{m} \binom{x + v_i + \dots + v_{i-r+1} - 1}{v_i}$$

$$\times \prod_{i \geq m+1} \binom{v_i + \dots + v_{i-r+1} - 1}{v_i}.$$

Corollary 6.6. *In the ring of formal power series,*

$$F_r(x_1, x_2, \dots) = \exp\Big(\sum_{\mathbf{v} \neq \mathbf{0}} \prod_i x_i^{v_i} \frac{1}{v_b} \prod_{i \geq b+1} \binom{v_i + v_{i-1} + \dots + v_{i-r+1} - 1}{v_i}\Big),$$

where $b = b(\mathbf{v}) = \min_{v_i > 0} i$.

Proof. By Corollary 6.4,

$$\log F_r(x_1, x_2, \dots) = \sum_{j=1}^{\infty} \log CF_r(-x_j, -x_{j+1}, \dots)$$

$$= \sum_{j=1}^{\infty} \log(1 + CF_r(-x_j, \dots) - 1) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(CF_r(-x_j, \dots) - 1)^k$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^{k+1}}{k}(CF_r(-x_j, ..))^i (-1)^{k-i}$$

$$= \sum_{j=1}^{\infty} \sum_{\substack{\mathbf{v} \\ b(\mathbf{v})=j}} \prod_i x_i^{v_i} \prod_{a \geq j+1} \binom{v_a + \dots + v_{a-r+1} - 1}{v_a}$$

$$(19) \qquad \times \sum_{k=1}^{\infty} \sum_i \binom{k}{i} \frac{(-1)^{k+1+k-i}}{k} \binom{i + v_j - 1}{v_j}$$

(by Theorem 6.5). Applying binomial inversion to the identity

$$\sum_k \binom{x}{k} \binom{n-1}{k-1} = \binom{x+n-1}{n},$$

the sum over k and i in (19) equals

$$\sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \binom{v_j - 1}{k-1} = \int_0^1 \sum_{k=1}^{v_j} (-1)^{k-1} x^{k-1} \binom{v_j - 1}{k-1} dx$$

$$= \int_0^1 (1-x)^{v_j} dx = 1/v_j$$

and the corollary follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Remark. The coefficient

$$\prod_{i \geq j+1} \binom{v_i + v_{i-1} + \ldots + v_{i-r+1} - 1}{v_i}$$

also has a combinatorial interpretation; it is the number of permutations (of the multiset having v_i copies of i) with no r-descents, and which end with j (the smallest element).

Example 6.7. MacMahon noted that the Rogers-Ramanujan identities are equivalent to

$$E_2(1, q, q^2, q^3, \ldots) = \frac{1}{1-q^1} \frac{1}{1-q^4} \frac{1}{1-q^6} \frac{1}{1-q^9} \frac{1}{1-q^{11}} \frac{1}{1-q^{14}} \cdots,$$

and

$$E_2(1, q^2, q^3, q^4, \ldots) = \frac{1}{1-q^2} \frac{1}{1-q^3} \frac{1}{1-q^7} \frac{1}{1-q^8} \frac{1}{1-q^{12}} \frac{1}{1-q^{13}} \cdots.$$

Taking the reciprocal of both sides of these equations, and using the combinatorial interpretation of the coefficients of $F_2(x_1, x_2, \ldots)$ (see the remark following the proof of Theorem 6.2), we get

$$(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots = \sum_{n \geq 0} q^n \sum_{\substack{\lambda \vdash n \\ \lambda_i \geq \lambda_{i+1}-1}} (-1)^{\#\text{parts}},$$

and

$$(1-q^2)(1-q^3)(1-q^7)(1-q^8)\cdots = \sum_{n \geq 0} q^n \sum_{\substack{\lambda \vdash n \\ \lambda_i \geq \lambda_{i+1}-1 \\ \lambda_i \geq 2}} (-1)^{\#\text{parts}}.$$

Here the sums are over "semi-partitions" λ of n (we allow consecutive parts to increase, but only by at most one), and "# parts" is the number of parts of λ.

7. Some Other Dirichlet Series

By definition, $\zeta_r(s) = F_r(1/p_1^s, 1/p_2^s, \ldots)$, and the combinatorial identities listed in section 2 are thus special cases of those in section 6. Given an arbitrary subset $\mathcal{D}$ of $\mathcal{D}_r$, let

$$\mu_{\mathcal{D}}(n) = \begin{cases} \mu(n), & \text{if } n \text{ is not divisible by any } d \in \mathcal{D}; \\ 0, & \text{else}, \end{cases}$$

and set

$$\zeta_{\mathcal{D}}(s) = \left(\sum_{n=1}^{\infty} \frac{\mu_{\mathcal{D}}(n)}{n^s}\right)^{-1}.$$

Then, by the same argument as in Corollary 2.2, Lemma 5.2 implies that (assuming the RH) $\zeta_{\mathcal{D}}(s)$ has an analytic continuation to the line $\sigma = 1/2$, and is zero free for $\sigma > 1/2$. The choice $\mathcal{D} = \mathcal{D}_r$ is not the only choice for which $\zeta_{\mathcal{D}}(s)$ has a product representation; for example, if $\mathcal{D} = \{q_{2i-1}q_{2i} \mid i \geq 1\}$ then

$$\zeta_{\mathcal{D}}(s) = \frac{1}{(1 - 2^{-s} - 3^{-s})} \frac{1}{(1 - 5^{-s} - 7^{-s})} \frac{1}{(1 - 11^{-s} - 13^{-s})} \cdots.$$

For another example, let $\mathcal{D} = \{p_i p_{i+1} \mid i \geq 1\} \cup \{p_{2i} p_{2i+2} \mid i \geq 1\}$. Then

$$\zeta_{\mathcal{D}}(s) = \prod_{\substack{i \geq 1 \\ i \text{ odd}}} \cfrac{1}{1 - p_i^{-s} - \cfrac{p_{i+1}^{-s}}{1 - p_{i+2}^{-s} - \cfrac{p_{i+3}^{-s}}{1 - p_{i+4}^{-s} - \cfrac{p_{i+5}^{-s}}{1 - \cdots}}}}$$

(continued fractions of this type involving powers of q instead of primes have applications to partition theory; see [All]).

Extension to negative r. The coefficient of $1/n^s$ in $\zeta_r(s)$ equals the number of permutations of the associated multiset having no r-descents. An r-descent is simply a value of i such that $\sigma_i > \sigma_{i+1} + r - 1$, and there is no problem in this definition if we allow $r < 1$. For example, we should define

$$\zeta_0(s) = \prod_p (1 + \frac{1}{p^s}),$$

since in order for a permutation to have no 0-descents, there must be an ascent of at least one at every spot, which is possible if and only if the multiset has distinct elements (i.e. n is squarefree). Note that $\zeta_0(s)$ factors as $\zeta(s)/\zeta(2s)$, which under the RH has all of its zeros on the line $\sigma = 1/2$. In general define

$$\zeta_r(s) = \sum_{n=1}^{\infty} \frac{|\mu_{1-r}(n)|}{n^s}, \qquad r < 1.$$

By (18) and Corollary 6.6 we can write

$$\zeta_r(s) = E_{1-r}(1/p_1^s, 1/p_2^s, \dots) \qquad (r < 1)$$

$$= F_{1-r}(-1/p_1^s, -1/p_2^s, \dots)^{-1}$$

$$= \exp\left(\sum_{n > 1} \frac{(-1)^{\nu(n)+1}}{n^s \nu_b} \prod_{j \geq b+1} \binom{\nu_j + \nu_{j-1} + \dots + \nu_{j-r+1} - 1}{\nu_j}\right).$$

Using this identity the proof of Theorem 4.4 can be extended to show that, given $\delta > 0$,

$$\zeta_r(s) = h_r(s) \prod_{k=1}^{N_r} \zeta(ks)^{a_k(r)}, \qquad r < 0,$$

where $h_r(s)$ has no poles (but may have zeros) in $\sigma > \delta$ and $N_r \to \infty$ as $\delta \to 0$. The integers $a_k(r)$ are positive for k odd and negative for k even, which can be used to show that the line $\sigma = 0$ forms a natural boundary for $\zeta_r(s)$, $r < 0$.

8. r-MULTIPLICATIVE FUNCTIONS

Although the coefficient of n^{-s} in $\zeta_r(s)$,

$$\prod_{i \geq 1} \binom{\nu_i + \nu_{i-1} + \dots + \nu_{i-r+1}}{\nu_i},$$

is not multiplicative, it does satisfy a certain generalized concept of multiplicativity. Call a function h r-multiplicative if $h(mn) = h(m)h(n)$ whenever $p_i \mid m, p_j \mid n \implies \mid j - i \mid \geq r$. The following properties are easily established:

Let h and x be two r-multiplicative functions. Then:

1) If $y(n) = \sum_{d \mid n} h(d)x(n/d)$, then y is r-multiplicative.

2) If $(\sum_{n \geq 1} h(n)/n^s)^{-1} = \sum_{n \geq 1} w(n)/n^s$, then w is r-multiplicative.

9. Some Questions for Future Research

1) Where are the poles of $\zeta_r(s)$ and what are their orders?

2) Does $\sum_{n \leq x} \mu_r(n) = O(x^{1/2+\epsilon})$ for some $r > 1$ imply $M(x) = O(x^{1/2+\epsilon})$?

3) One of the interesting things about $\zeta_2(s)$ is that it admits a "natural" q-version. By replacing all binomial coefficients by q-binomial coefficients we have a function which equals $\zeta_2(s)$ for $q=1$ and $\zeta(s)$ for $q = 0$. Where are the zeros of this function for $0 < q < 1$? (A different q-version of $\zeta(s)$, constructed by interpolating Carlitz's q-Bernoulli numbers, has been introduced by Satoh [Sat].)

4) Carlitz [Car] studied the polynomials $H_n(\lambda)$ defined by

$$(20) \qquad \frac{1 - \lambda}{\zeta(s) - \lambda} = \sum_{n=1}^{\infty} \frac{H_n(\lambda)}{n^s},$$

and Dillon and Roselle [D-R] noted the coefficients of $H_n(\lambda)$ are the Simon Newcomb numbers $N_k(\nu(n))$ described in section 6. In the same paper Carlitz introduced generalized Bernoulli numbers $\beta(n)$ defined by

$$(21) \qquad \frac{\log \zeta(s)}{\zeta(s) - 1} = \sum_{n=1}^{\infty} \frac{\beta(n)}{n^s}$$

$(\beta(p_1 p_2 \cdots p_k) = B_k$, the kth Bernoulli number). If we replace $\zeta(s)$ by $\zeta_r(s)$ in (20) we generate the numbers $N_{k,r}(\nu(n))$ from section 6. If we make the same substitution in (21), what properties will the corresponding numbers $\beta_r(n)$ have?

Acknowledgement

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